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Parametric Robust $\mathcal{H}_2$ Control Design Using Iterative Linear Matrix Inequalities Synthesis

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I. Introduction

CONTROL design to satisfy robust performance objectives with real parametric uncertainty has recently attracted much attention in the controls community. One of the many proposed techniques is Popov controller synthesis, which captures the system uncertainty as a sector-bounded nonlinearity. A Lyapunov function representation of the Popov analysis test can also be combined with an $\mathcal{H}_2$ cost function to provide a bound on the robust performance. Designing a controller to optimize this bound provides a synthesis tool that guarantees robust stability and performance. Several previous investigations[1–4] have been performed to develop solutions to this control design problem. However, until recently, these approaches were based on the numerical optimization using quasi-Newton search algorithms to solve the necessary conditions that must be analytically derived from the performance objective and constraint equations. This previous work clearly demonstrated that Popov controller synthesis can be used to design very robust controllers, but there exist considerable drawbacks to this approach in practice, including substantial effort required to compute the analytic gradients.[2–5] Moreover, the computational effort required by the gradient search algorithm is intensive, and the rate of convergence for large-order problems is typically quite slow.[2,3,5] This Note introduces a new combined analysis and synthesis procedure that potentially eliminates many of the numerical and implementation difficulties of the quasi-Newton approach. The resulting algorithm is essentially the $\mathcal{H}_2$ equivalent of the D–K iteration used to design parametric robust $\mathcal{H}_\infty$ controllers.

This new approach is based on linear matrix inequalities (LMIs) that Boyd et al. (Ref. 6, Sec. 8.1) have investigated for the Popov analysis problem. The direct extension of this LMI analysis to the controller synthesis problem results in bilinear matrix inequalities (BMIs), which currently are difficult to solve directly. El Ghaoui and Balakrishnan[7] proposed an iterative procedure to solve these BMI problems using a two-stage optimization process. During each phase of this iteration, some of the design variables in the BMIs are fixed, leading to LMIs in the remaining variables. This technique has been shown to work well on simple examples,[7] but on more complicated objectives, such as control design to minimize an $\mathcal{H}_2$ cost function, experience has shown that the technique converges very slowly, if at all. The robust controller synthesis algorithm was recently improved by El Ghaoui and Folcher leading to a more systematic design approach for systems with unstructured uncertainty[8] and with structured uncertainty.[9] However, a single quadratic Lyapunov function is used in Refs. 8 and 9 that can be a source of significant conservatism in the robust control design for systems with real parametric uncertainty.

The primary purpose of this paper is to demonstrate a new numerical algorithm for robust $\mathcal{H}_2$ control design. Our design approach combines the Popov stability analysis and worst-case $\mathcal{H}_2$ performance bounds of Lur'e systems. We then apply LMI synthesis tools to solve the robust controller design problem. Numerical results on several standard benchmark problems are presented to show the robustness of the resulting controllers and the rapid convergence of the iterative algorithm.

II. Problem Statement

We consider an LTI system (nominal system) subject to sector bounded nonlinear uncertainty, that is, a Lur'e system, described by

$$\begin{aligned}
\dot{x} &= Ax + B_p p + B_w \boldsymbol{w} + B_u u \\
q &= C_q x + D_{qp} p + D_{qw} \boldsymbol{w} + D_{qu} u \\
z &= C_z x + D_{zp} p + D_{zw} \boldsymbol{w} + D_{zu} u \\
y &= C_y x + D_{yp} p + D_{yw} \boldsymbol{w} + D_{yu} u, \qquad p = \phi(q)
\end{aligned} \tag{1}$$

where $\boldsymbol{x} : \boldsymbol{R}_+ \to \boldsymbol{R}^n$ is the state, $\boldsymbol{u} : \boldsymbol{R}_+ \to \boldsymbol{R}^{n_u}$ is the control input, $\boldsymbol{w} : \boldsymbol{R}_+ \to \boldsymbol{R}^{n_w}$ is the disturbance input, $\boldsymbol{y} : \boldsymbol{R}_+ \to \boldsymbol{R}^{n_y}$ is the measured output, $\boldsymbol{z} : \boldsymbol{R}_+ \to \boldsymbol{R}^{n_z}$ is the performance output, $\boldsymbol{q} : \boldsymbol{R}_+ \to \boldsymbol{R}^{n_p}$ and $\boldsymbol{p} : \boldsymbol{R}_+ \to \boldsymbol{R}^{n_p}$ are the input/output of the nonlinear uncertainty ϕ. The nonlinear perturbation ϕ is assumed to satisfy the sector bound [0, 1]; that is, $\phi \in \Phi$ where $\Phi := \{\phi : \boldsymbol{R}^{n_p} \to \boldsymbol{R}^{n_p},\ \phi(q) = [\phi_1(q_1), \ldots, \phi_{n_p}(q_{n_p})]^T$ and $0 \le \phi_i(\sigma)/\sigma \le 1,\ \forall i = 1, \ldots, n_p\}$. As discussed in Ref. 6 (p. 129), loop transformations can be used to handle more general sector conditions $\alpha_i \le \phi_i(\sigma)/\sigma \le \beta_i$. The description of the Lur'e system also includes an important class of uncertain systems considered in Refs. 1–4. These systems are described by $\dot{x} = (A + \Delta A)x + B_w w + B_u u$, $\Delta A \in \mathcal{U}$, where $\mathcal{U} := \{\Delta A \in \boldsymbol{R}^{n \times n} : \Delta A = B_p D C_q,\ \ D = \mathrm{diag}(\delta_1, \ldots, \delta_{n_p}),\ \delta_i \in [0, 1],\ \forall i = 1, \ldots, n_p\}$. In control theory, this is referred to as the system subject to real parametric uncertainty. This special case of the Lur'e system (1) occurs when the functions ϕ_i are linear, that is, $\phi_i(\sigma) = \delta_i \sigma$, where $\delta_i \in [0,\ 1]$, $\forall i = 1, \ldots, n_p$. For a well-posed nature, we assume that D_{zw} is identically zero, and to significantly simplify the analysis and synthesis, we assume D_{zp}, D_{qp}, and D_{qw} are identically zero. Also note that it is essential to have D_{qu} equal zero to use this control design control procedure. Finally, let $U \in \boldsymbol{R}^{n \times p}$. $U_\perp$ is defined as an orthogonal complement of U, that is, $U^T U_\perp = 0$ and $[U\ \ U_\perp]$ is of maximum rank.

A. Popov Controller Synthesis

Based on the discussion of the worst-case performance of nonlinear systems in Ref. 10, we define the worst-case $\mathcal{H}_2$ performance, $\mathcal{J}$, for Eq. (1) as

$$\mathcal{J} := \sup \sum_{i=1}^{n_w} \int_0^\infty z_i(t)^T z_i(t)\, \mathrm{d}t$$

where the supremum is taken over all nonzero output trajectories $\{z_1(t), \ldots, z_{n_w}(t)\}$ of the nonlinear system (1) starting from $\boldsymbol{x}(0) = 0$ and subject to the impulse disturbances $\{\delta w_1, \ldots, \delta w_{n_w}\}$

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(the impulses in the direction of the vectors $w_1, \ldots, w_{n_w}$). Note that an impulse relates to a nonzero initial condition in a sense that an impulse causes an equivalent effect as a nonzero initial condition. Hence, $\{z_1(t), \ldots, z_{n_w}(t)\}$ are equivalently the zero-input responses due to the initial conditions $x_i(0) = B_w e_i$, $i = 1, \ldots, n_w$, where $\{e_1, \ldots, e_{n_w}\}$ is a basis of $\boldsymbol{R}^{n_w}$. Although $\mathcal{J}$ is difficult to evaluate, we can readily compute its upper bound, which is independent of the particular basis and realization[10] and can be formulated as a convex optimization problem.[1,6]

Theorem 1 (Refs. 1 and 6): If there exists a Lyapunov function

$$V(\boldsymbol{x}) = \boldsymbol{x}^T P \boldsymbol{x} + 2 \sum_{i=1}^{n_p} \lambda_i \int_0^{C_{i,q}\boldsymbol{x}} \phi_i(\sigma)\, \mathrm{d}\sigma \tag{2}$$

where $C_{i,q}$ denotes the ith row of C_q, and $\Lambda := \mathrm{diag}(\lambda_1, \ldots, \lambda_{n_p}) \geq 0$, and $T := \mathrm{diag}(\tau_1, \ldots, \tau_{n_p}) \geq 0$, satisfying

$$\begin{bmatrix} A^T P + PA + C_z^T C_z & PB_p + A^T C_q^T \Lambda + C_q^T T \\ B_p^T P + \Lambda C_q A + T C_q & \Lambda C_q B_p + B_p^T C_q^T \Lambda - 2T \end{bmatrix} \leq 0 \tag{3}$$

then the upper bound of $\mathcal{J}$ is finite and can be computed by minimizing $\mathrm{tr}\, B_w^T [P + C_q^T \Lambda C_q] B_w$, over the variables P, Λ, and T, that is,

$$\begin{aligned} &\text{minimize} \quad \mathrm{tr}\, B_w^T \left[P + C_q^T \Lambda C_q\right] B_w \\ &\text{subject to Eq. (3)}, \quad P > 0, \quad \Lambda \geq 0, \quad T \geq 0 \end{aligned} \tag{4}$$

Proof: See Ref. 6, pp. 121–122 and Ref. 1. We apply the analysis result in Theorem 1 to formulate the Popov controller synthesis problem. In particular, our goal is to find a strictly proper full-order LTI controller that minimizes the upper bound on the worst-case $\mathcal{H}_2$ performance of the closed-loop Lur'e system. The controller is

$$\dot{\boldsymbol{x}}_c = A_c \boldsymbol{x}_c + B_c \boldsymbol{y}, \qquad \boldsymbol{u} = C_c \boldsymbol{x}_c \tag{5}$$

where $\boldsymbol{x}_c : \boldsymbol{R}_+ \rightarrow \boldsymbol{R}^n$ is the controller state and A_c, B_c, and C_c are constant matrices of appropriate size. The closed-loop system of the Lur'e system (1) and the LTI controller (5) have dynamics

$$\begin{aligned} \dot{\tilde{\boldsymbol{x}}} &= \tilde{A}\tilde{\boldsymbol{x}} + \tilde{B}_p \boldsymbol{p} + \tilde{B}_w \boldsymbol{w}, \qquad & \boldsymbol{q} &= \tilde{C}_q \tilde{\boldsymbol{x}} + \tilde{D}_{qp}\boldsymbol{p} + \tilde{D}_{qw}\boldsymbol{w} \\ \boldsymbol{z} &= \tilde{C}_z \tilde{\boldsymbol{x}} + \tilde{D}_{zp}\boldsymbol{p} + \tilde{D}_{zw}\boldsymbol{w}, \qquad & \boldsymbol{p} &= \phi(\boldsymbol{q}) \end{aligned} \tag{6}$$

where

$$\left[\begin{array}{c|c|c} \tilde{A} & \tilde{B}_p & \tilde{B}_w \\ \hline \tilde{C}_q & \tilde{D}_{qp} & \tilde{D}_{qw} \\ \hline \tilde{C}_z & \tilde{D}_{zp} & \tilde{D}_{zw} \end{array}\right] = \left[\begin{array}{cc|c|c} A & B_u C_c & B_p & B_w \\ B_c C_y & A_c + B_c D_{yu} C_c & B_c D_{yp} & B_c D_{yw} \\ \hline C_q & D_{qu} C_c & D_{qp} & D_{qw} \\ \hline C_z & D_{zu} C_c & D_{zp} & D_{zw} \end{array}\right]$$

and $\tilde{\boldsymbol{x}} = [\boldsymbol{x}^T \ \ \boldsymbol{x}_c^T]^T$. It is straightforward to compute the upper bound of the worst-case $\mathcal{H}_2$ performance of the closed-loop Lur'e system (6) by solving the following optimization problem:

$$\begin{aligned} &\text{minimize } \mathrm{tr}\, \tilde{B}_w^T \left[\tilde{P} + \tilde{C}_q^T \Lambda \tilde{C}_q\right] \tilde{B}_w \\ &\text{subject to} \\ &\begin{bmatrix} \tilde{A}^T \tilde{P} + \tilde{P}\tilde{A} + \tilde{C}_z^T \tilde{C}_z & \tilde{P}\tilde{B}_p + \tilde{A}^T \tilde{C}_q^T \Lambda + \tilde{C}_q^T T \\ \tilde{B}_p^T \tilde{P} + \Lambda \tilde{C}_q \tilde{A} + T\tilde{C}_q & \Lambda \tilde{C}_q \tilde{B}_p + \tilde{B}_p^T \tilde{C}_q^T \Lambda - 2T \end{bmatrix} \leq 0 \\ &\tilde{P} > 0, \quad \Lambda \geq 0, \quad T \geq 0 \end{aligned} \tag{7}$$

This $\mathcal{H}_2$ cost overbound is a function of $(\tilde{P}, \Lambda, T)$ resulting from the worst-case $\mathcal{H}_2$ performance analysis and (A_c, B_c, C_c), which are the compensator parameters. Note also that the first constraint of Eq. (7) is bilinear matrix inequality because there are product terms involving the design parameters $(\tilde{P}, \Lambda, T)$ and (A_c, B_c, C_c).

III. Design Procedure

The robust performance analysis in Refs. 8 and 9 uses a quadratic Lyapunov function, that is, $\boldsymbol{x}^T P \boldsymbol{x}$, which can be conservative for systems with real parametric uncertainty. The following development parallels Refs. 8 and 9 with the extension of using the quadratic plus integral Lyapunov function (2) to reduce this conservatism. There are two key steps in developing the algorithm.

A. Controller Elimination

Because A_c only appears in the first constraint of Eq. (7), we can reduce the number of variables in the design problem by eliminating A_c. To show this, we notice that $\tilde{A} = \tilde{A}_0 + \tilde{J} A_c \tilde{J}^T$, where

$$\tilde{A}_0 := \begin{bmatrix} A & B_u C_c \\ B_c C_y & B_c D_{yu} C_c \end{bmatrix}, \qquad \tilde{J} := \begin{bmatrix} 0 \\ I \end{bmatrix}, \qquad \tilde{J}_\perp := \begin{bmatrix} I \\ 0 \end{bmatrix}$$

and rewrite the first constraint of Eq. (7) as

$$\tilde{G} + V A_c^T U^T + U A_c V^T < 0 \tag{8}$$

where $\tilde{G}$, V, and U are

$$\tilde{G} := \begin{bmatrix} \tilde{A}_0^T \tilde{P} + \tilde{P}\tilde{A}_0 + \tilde{C}_z^T \tilde{C}_z & \tilde{P}\tilde{B}_p + \tilde{A}_0^T \tilde{C}_q^T \Lambda + \tilde{C}_q^T T \\ (\tilde{P}\tilde{B}_p + \tilde{A}_0^T \tilde{C}_q^T \Lambda + \tilde{C}_q^T T)^T & \Lambda \tilde{C}_q \tilde{B}_p + (\Lambda \tilde{C}_q \tilde{B}_p)^T - 2T \end{bmatrix}$$

$$V := \begin{bmatrix} \tilde{J} \\ 0 \end{bmatrix}, \qquad U := \begin{bmatrix} \tilde{P}\tilde{J} \\ 0 \end{bmatrix}$$

By application of the elimination lemma (Ref. 6, p. 32), it follows that the first constraint of Eq. (7) holds if and only if

$$V_\perp^T \tilde{G} V_\perp < 0, \qquad U_\perp^T \tilde{G} U_\perp < 0 \tag{9}$$

To proceed, we partition $\tilde{P}$ and its inverse $\tilde{Q}$ as

$$\tilde{P} = \begin{bmatrix} P & M \\ M^T & R \end{bmatrix}, \qquad \tilde{Q} = \tilde{P}^{-1} = \begin{bmatrix} Q & N \\ N^T & S \end{bmatrix} \tag{10}$$

where P and Q are symmetric matrices in $\boldsymbol{R}^{n \times n}$, $R = M^T(P - Q^{-1})^{-1}M$, $S = N^T(Q - P^{-1})^{-1}N$, and $N = (I - QP)M^{-T}$. We define $Y := C_c N^T$ and $Z := M B_c$. After some algebra, constraints (9) are equivalent to

$$\begin{bmatrix} PA + ZC_y + (PA + ZC_y)^T + C_z^T C_z & PB_p + ZD_{yp} + A^T C_q^T \Lambda + C_q^T T \\ (PB_p + ZD_{yp} + A^T C_q^T \Lambda + C_q^T T)^T & \Lambda C_q B_p + (\Lambda C_q B_p)^T - 2T \end{bmatrix} < 0$$

$$\begin{bmatrix} AQ + B_u Y + (AQ + B_u Y)^T & B_p + (AQ + B_u Y)^T C_q^T \Lambda + QC_q^T T & (C_z Q + D_{zu} Y)^T \\ (B_p + (AQ + B_u Y)^T C_q^T \Lambda + QC_q^T T)^T & \Lambda C_q B_p + (\Lambda C_q B_p)^T - 2T & 0 \\ C_z Q + D_{zu} Y & 0 & -I \end{bmatrix} < 0 \tag{11}$$

By the completion lemma,[11] for every $Q > 0$, we have that $P \geq Q^{-1}$. Moreover, if $\tilde{P} > 0$ and $\tilde{P}\tilde{Q} = I$, then

$$\begin{bmatrix} P & I \\ I & Q \end{bmatrix} \geq 0 \tag{12}$$

By restricting Eq. (12) to be positive definite, we are effectively searching for full-order controllers, that is, of order n.[8,9] Note that the second inequality in Eq. (11) is bilinear, that is, there are product terms involving (Q, Y) and (Λ, T). If Λ and T are fixed, then Eqs. (11) are LMIs in Q and Y. Similarly, if Q and Y are fixed, then Eqs. (11) are LMIs in Λ and T. As discussed in Refs. 8 and 9, the trace condition appearing as the performance index of Eq. (7), that is, $\operatorname{tr} \tilde{B}_w^T(\tilde{P} + \tilde{C}_q^T \Lambda \tilde{C}_q)\tilde{B}_w$, is equivalent to

$$\operatorname{tr}\begin{bmatrix} B_w \\ D_{yw} \end{bmatrix}^T \begin{bmatrix} P + C_q^T \Lambda C_q & Z \\ Z^T & X \end{bmatrix} \begin{bmatrix} B_w \\ D_{yw} \end{bmatrix} \tag{13}$$

and the existence of a symmetric matrix $X \in \boldsymbol{R}^{n_y \times n_y}$ such that

$$\begin{bmatrix} X & Z^T & 0 \\ Z & P & I \\ 0 & I & Q \end{bmatrix} > 0 \tag{14}$$

which implies $\tilde{P} > 0$. In summary, after eliminating A_c from the robust performance constraint (11), the Popov controller synthesis problem (7) is equivalent to

$$\begin{aligned} &\text{minimize Eq. (13)} \\ &\text{subject to Eqs. (11) and (14)}, \quad \Lambda \geq 0, \quad T \geq 0 \end{aligned} \tag{15}$$

where unknown variables include P, Z, Q, Y, X, Λ, and T.

B. Controller Reconstruction

Given that there exist P, Z, Q, Y, X, Λ, and T satisfying Eq. (15), we can construct a controller as follows. We first construct $\tilde{P}$ such that the first constraint of Eq. (7) holds. Note that $\tilde{P}$ is parameterized by Eq. (10), where M is an arbitrary invertible matrix. Because M corresponds to a change of coordinates in the controller states $\boldsymbol{x}_c$, the choice of M has no effect on the controller transfer function.[8,9] Then, we compute B_c and C_c, which are parameterized by $B_c = M^{-1}Z$ and $C_c = Y(I - PQ)^{-1}M$. Last, with $\tilde{P}$, Λ, T, B_c, and C_c determined, it suffices to find A_c satisfying Eq. (8), which can then be formulated as an LMI feasibility problem in A_c. As discussed in Refs. 8 and 9, it is also feasible to explicitly compute an analytical solution of A_c satisfying Eq. (8).

C. Algorithm

It has already been shown that BMI problems are NP-hard, and it is thought to be rather unlikely that there is a polynomial time algorithm to compute the optimal solutions. Our approach to solving the nonconvex optimization problem is based on an iterative procedure. The proposed algorithm, which we also call the V–K iteration, alternates between three different LMI problems, that is, Eq. (7) with fixed compensator parameters (A_c, B_c, and C_c), Eq. (15) with fixed multiplier parameters (Λ, T), and Eq. (8). (The name V–K iteration is chosen to be historically consistent with Ref. 7, but key differences exist between the two implementations.[12]) The algorithm to design a family of controllers with increasing robustness is briefly summarized as follows.

1) Initialize the nonlinear sector bound to be zero (a nominal system) and design the controller via Linear Quadratic Gaussian (LQG) or any other simple robust design techniques.
2) Initialize (Λ, T) by solving Eq. (7) where (A_c, B_c, C_c) are fixed.
3) Repeat{ [Outer Loop]
   Repeat { [Inner Loop]
   a) Solve the optimization problem (15), i.e., solving for (P, Z, Q, Y, X) where Λ and T are fixed. Then compute $\tilde{P}$, B_c, and C_c using the completion lemma (Ref. 11).
   b) Compute A_c by solving a feasibility LMI problem (8).
   c) Compute (Λ, T) by solving Eq. (7) where (A_c, B_c, C_c) are fixed.
   } [Inner Loop] Until stopping criterion satisfied.
   Increase the nonlinear sector bound to the next desired size and initialize Λ and T by the most recent values.
   } [Outer Loop] Until the desired robustness is achieved or the problem is infeasible.

The procedure of alternating between the LMI problems is an iterative approach for solving non-convex optimization problems. Although it is generally not guaranteed to converge globally, our experience is that this algorithm efficiently converges to a local optimum. Because this parametric robust $\mathcal{H}_2$ algorithm iterates between solving for the optimal controller and multiplier parameters, our approach is essentially the $\mathcal{H}_2$ equivalent of the D–K iteration used to design parametric robust $\mathcal{H}_\infty$ controllers.

Note that a similar approach has been reported concurrently with Ref. 12 by Yang et al.[13] However, a major difference exists in the controller elimination step of these two references. In Ref. 13, the compensator variable B_c is fixed; A_c and C_c are then removed from the BMI problem (7). In our approach, only A_c is eliminated from the first constraint of Eq. (7), but the resulting optimization formulation shown in Eq. (11) consists of a smaller number of unknowns (for example, we solve directly for X, Y, and Z instead of the full block of $\tilde{P}$) and a smaller dimension of the matrix inequalities, in particular Eq. (14) and the equivalent in theorem 4 of Ref. 13. This is an important distinction because of the memory limitations often associated with the current forms of the LMI solver and interface.[14] Moreover, because of the simple forms of U and V, and their orthogonal complements ($U_\perp$ and $V_\perp$) in Eq. (8), the elimination step in our algorithm results in a more concise representation of Eq. (11)

IV. Numerical Examples

A. Three Mass–Spring System with One Uncertainty

We investigate the new design approach by comparing our controllers to those obtained for a three mass–spring system benchmark problem selected from Ref. 4. The system consists of three masses connected by two springs, in which the spring uncertainty between the second and third masses is written as $k_2 = k_{2,\text{nom}}(1 + \delta)$, where $k_{2,\text{nom}}$ is the nominal value and the uncertainty is captured by $\delta \in \boldsymbol{R}$. The system parameters are $m_1 = m_2 = m_3 = 1$, $k_1 = 1$, and $k_{2,\text{nom}} = 1$. To apply this Popov analysis and synthesis, we effectively approximate the uncertainty in the spring stiffness as $k_2(x) = k_{2,\text{nom}}[x + \sigma\phi(x)]$, where $\sigma > 0$ is a measure of the relative guaranteed uncertainty bound, and $\phi(x)$ is a $[-1, 1]$ sector-bounded memoryless nonlinear function of the spring displacement x. Thus, this approach can be used to treat sector-bounded nonlinear uncertainties or approximate linear uncertainties.

Several controllers were designed using the LMI synthesis algorithm presented in Sec. III for various values of σ. Note that for this sample problem, each outer-loop iteration of the algorithm in Sec. III.C required less than 1 min to execute on a Sun-Sparc 20/60. We directly compare our Popov compensators to the robust controllers in Ref. 4 obtained via quasi-Newton methods. The robust performance result in Fig. 1 plots the normalized $\mathcal{H}_2$ cost (i.e., the $\mathcal{H}_2$ cost normalized by the $\mathcal{H}_2$ cost for the nominal system with the LQG design) as a function of changes in the spring stiffness k_2. The vertical asymptotes in Fig. 1 correspond to the achieved stability boundaries for each compensator. Seven controllers are compared: one reference LQG design, three Popov designs from Ref. 4 designed at $\sigma = [0.05, 0.11, 0.14]$, and three Popov designs using LMI synthesis designed at the same guaranteed robustness bounds. We also compared the frequency response of these seven controllers and the s-plane location of the compensator poles and zeros. The graphical comparison is omitted because of space limitations, but can be found in Ref. 12. These robust performance results confirm that the two completely independent design methods give identical robust performance curves.

Consider the convergence of the Popov controller synthesis algorithm with $\sigma = 0.05$. The mesh plot shown in Fig. 2 illustrates the worst-case $\mathcal{H}_2$ cost overbounds for a selected set of multiplier

Table 1 Summary of the *V–K* iteration of the Popov controller designed for the three mass–spring system with ± 5% changes of the second spring constant

Iteration	Upper bound of $\mathcal{H}_2$ cost	% error in $\mathcal{H}_2$ cost	Multiplier parameters Λ	Multiplier parameters T
Initial	47.0867	168.6982	45.3581	91.4197
1	17.6054	0.4643	1.4370	3.1011
2	17.5303	0.0359	1.4779	2.5587
3	17.5246	0.0032	1.4741	2.4267
Optimal	17.5240	0	1.4741	2.3781

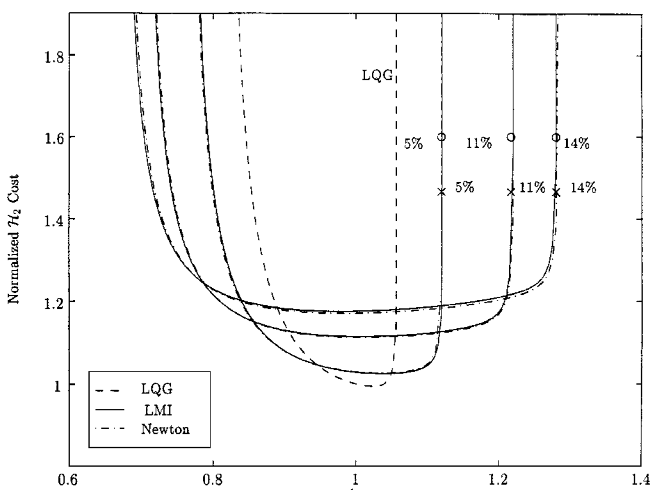

Fig. 1 Robust performance comparison of Popov controllers designed using LMI synthesis and quasi-Newton methods[4]; normalized $\mathcal{H}_2$ cost for the LQG design given as a reference.

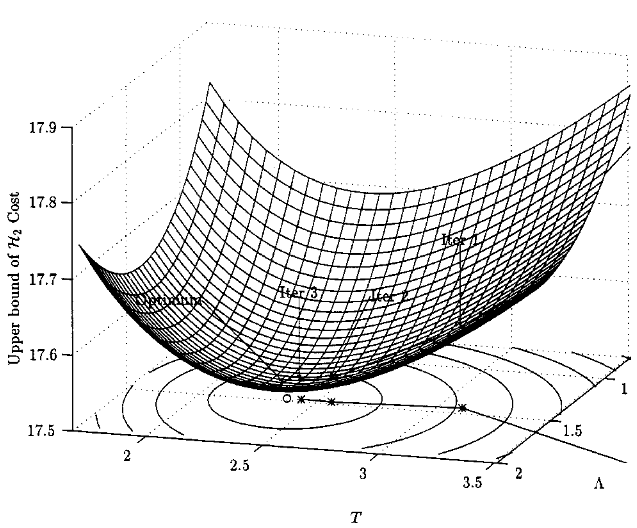

Fig. 2 Mesh plot of the $\mathcal{H}_2$ cost overbounds as a function of the multiplier parameters (Λ and T) after a refinement of the multiplier space; solid line is the path connecting the multiplier of each iteration marked by $*$.

values. Note that the plot only displays a narrow region of the entire space searched. The first surprising observation is that there is only one local minimum over the entire grid. Furthermore, the worst-case $\mathcal{H}_2$ cost overbound behaves like a convex function in this region of the multiplier parameters. In three dimensions, the cost surface shown in Fig. 2 looks like a smooth paraboloid that has a local minimum at $\Lambda = 1.474$ and $T = 2.378$.

The result after each iteration is marked by an asterisk in Fig. 2 (also see Table 1). The optimal Popov multiplier computed by the exhaustive search is marked by $\circ$. The initial multiplier lies far outside the region of the mesh plot. However, after the first iteration, the algorithm results in a multiplier for which the worst-case $\mathcal{H}_2$ cost overbound is within 0.5% of the optimal value. The algorithm

Table 2 Summary of the *V–K* iteration for the Popov controller synthesis for the three mass–spring system with ± 5% changes of both spring constants

Iteration	Upper bound of $\mathcal{H}_2$ cost	% error in $\mathcal{H}_2$ cost	Multiplier parameters $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2)$	Multiplier parameters $T = \mathrm{diag}(\tau_1, \tau_2)$
Initial	19.7722	0.7858	1.4209, 1.3155	2.3825, 1.8718
1	19.6497	0.1617	1.7352, 1.6280	2.4688, 1.9359
2	19.6251	0.0361	1.8826, 1.7836	2.4989, 1.9661
3	19.6196	0.0083	1.9536, 1.8593	2.5116, 1.9776
Optimal	19.6180	0	2.0126, 1.9227	2.5200, 1.9848

Table 3 Summary of the *V–K* iteration of the Popov controller designed for the Bernoulli–Euler beam with ± 2% guaranteed stability bounds

Iteration	Upper bound of $\mathcal{H}_2$ cost	% error in $\mathcal{H}_2$ cost	Multiplier parameters Λ	Multiplier parameters T
Initial	0.4034	263.6914	0.0751	1.0696
1	0.1121	1.1062	0.0005	0.0070
2	0.1110	0.0385	0.0008	0.0090
3	0.1109	0.0018	0.0008	0.0098
Optimal	0.1109	0	0.0008	0.0099

stops after the third iteration, which corresponds to a worst-case $\mathcal{H}_2$ cost overbound within 0.01% of the optimal value. The results clearly show that in this case, the algorithm converges to the optimal solution over the region of interest.

B. Three Mass–Spring System with Two Uncertainties

To further demonstrate the algorithm performance, we consider the three mass–spring system with uncertainty in both springs ($\pm 5\%$ guaranteed bounds). As before, we compute the Popov $\mathcal{H}_2$ controller using the *V–K* iteration. With a stopping criterion of 1% accuracy, three iterations are required to solve for this robust controller. Table 2 summarizes the history of the *V–K* iteration for the case with two uncertainties. Of course, there is no convenient visualization tool for the results of the exhaustive search in this case. As before, the overbound of the worst-case $\mathcal{H}_2$ performance is monotonically decreasing during the iterative solution. Furthermore, at the third iteration, the algorithm yields a multiplier for which the worst-case $\mathcal{H}_2$ cost overbound is within 0.01% of the optimal value. Note, however, that the multiplier parameters from this *V–K* iteration are not as close to the optimal ones as they were in the single uncertainty case.

C. Bernoulli–Euler Beam

To further test the proposed algorithm, we designed Popov controllers for a second standard benchmark problem.[15] The system is a cantilevered Bernoulli–Euler beam with unit length and mass density and stiffness scaled so that the fundamental frequency is 1 rad/s. The beam dynamics were modeled with four modes ($\omega_1 = 1$, $\omega_2 = 6.27$, $\omega_3 = 17.55$, and $\omega_4 = 34.39$ rad/s and damping $\zeta = 0.01$). The disturbance input, control input, sensor output, and performance output are all collocated at the tip of the beam, and the frequency of the third mode of the system is considered to be uncertain. With $\pm 5\%$ shifts in the modal frequency, there are substantial variations in the system gain and phase in the 17–25 rad/s frequency range.[15] The performance plots for three Popov controllers with ± 2, ± 4, $\pm 6\%$ guaranteed stability bounds are shown in Ref. 12.

The convergence is analyzed for the case with $\pm 2\%$ guaranteed stability bounds. The exhaustive procedure was used to compute the optimal solution (see Table 3). As in the earlier examples, there is only one local minimum over the entire grid, and the surface of the worst-case $\mathcal{H}_2$ cost behaves like a convex function for values of the multiplier parameters near this minimum. After the first iteration, the algorithm results in a multiplier for which the worst-case $\mathcal{H}_2$ cost overbound is within 1.2% of the optimal value. The algorithm stops after the third iteration, which corresponds to a worst-case $\mathcal{H}_2$ cost overbound within 0.01% of the optimal value. Reference 16 discusses an extension of the algorithm.

V. Conclusions

The numerical performance of a new LMI-based design technique for robust $\mathcal{H}_2$ controllers is analyzed for systems with real parametric uncertainty. This technique iterates between solving 1) LMIs for the analysis parameters with the controller values fixed and 2) LMIs for the controller values with the multiplier parameters fixed. The approach is, thus, quite similar to the D–K iteration typically used for μ synthesis. The results from this new algorithm were validated using a previously published benchmark example. Note that, from previous experience with both gradient optimization and LMI synthesis, an important potential advantage of the LMI approach is the low implementation overhead associated with this optimization, especially given the simplicity of current semidefinite programming interfaces.[14] This advantage also greatly simplifies the extension of the algorithm to include more general stability multipliers, as discussed in Ref. 16. Because controller synthesis is a nonconvex optimization problem, this iterative algorithm is generally not guaranteed to converge globally. However, for the numerical results presented on two standard benchmark problems, the synthesis algorithm is shown to rapidly converge close to the local optimal solution (typically in 3–4 iterations).

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New Class of Intermediate-Thrust Arcs for Trajectories in Newtonian Gravitational Fields

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Introduction

AS is known, the optimal trajectories of a rocket moving with constant exhaust velocity and limited mass-flow rate in a Newtonian gravitational field may consist of arcs of null thrust (NT), intermediate thrust (IT), and maximum thrust (MT).[1] The IT arcs represent singular arcs, and their appearance is a degenerate case with added difficulties in determining the controls.[2] All analytical solutions for the IT arcs can be divided into two classes: solutions for cases of free-and fixed-time problems. In the first case, Lawden's spiral,[1,2] spiral-shaped trajectories[3] and two classes of alternative spirals[4] in which the expression for the radius vector is the same as for Lawden's spiral are known. For the fixed-time problems some spiral,[4,5] circular,[6,7] and spherical trajectories[6,8] were found. It was shown that Lawden's spiral is nonoptimal.[4,5,9,10] Other solutions, taking into account the minimizing functional, do not satisfy the transversality condition[3] or Robbins's condition.[4,6–9] However, some spirals can satisfy these conditions and may be used for solving the interorbital transfer problem.[4] Although all of these results represent some progress in solving the problem, questions about existence of other solutions to IT arcs, their optimality, applicability, etc. have not been thoroughly studied. The analysis of known works shows that to verify optimality and applicability of IT arcs for the specific problem, it is necessary to investigate them for conditions of existence, optimality, transversality, and continuity at the switching points. The present work is devoted to developing this point of view on the basis of new spiral trajectories for the IT arcs.

Statement of the Problem

The equations of motion in the Newtonian field may be given in the form[1]

$$\dot{\boldsymbol{v}} = (cm/M)\boldsymbol{u} - (\mu/r^3)\boldsymbol{r}, \qquad \dot{\boldsymbol{r}} = \boldsymbol{v}, \qquad \dot{M} = -m$$

where $\boldsymbol{r} = (r, 0, 0)$ is the radius vector of the spacecraft with origin at the attracting center, $\boldsymbol{v} = (v_1, v_2, v_3)$ is the velocity vector, $\boldsymbol{u} = (u_1, u_2, u_3)$ is a unit thrust vector, M is the mass of the spacecraft, m $(0 \le m \le \bar{m})$ is the mass-flow rate, and c is the exhaust velocity. Here the components of all vectors are given in a spherical coordinate system (r, θ, and δ) with the origin at the attracting center.[7] The known theory of optimum trajectories does not allow use of constraints in the form of inequalities and, therefore, we transform the inequality for the m to an equality via introduction of a slack variable.[1] For this it is required that m satisfies the equality $m(\bar{m} - m) - \alpha^2 = 0$. Additionally, for the components of the thrust vector the following equality will hold: $u_1^2 + u_2^2 + u_3^2 = 1$. The m, α, and u_i $(i = 1, 2, 3)$ are piecewise continuous control functions. For simplification, all variables are denoted by x_i $(i = 1, \ldots, 7)$, that is, the components of $\boldsymbol{v}$ are denoted by x_1, x_2, and x_3; the components of $\boldsymbol{r}$ are denoted by x_4, x_5, and x_6; and the rocket mass is denoted by x_7. It is assumed that the initial conditions $x_j = x_{j0}$ $(j = 1, \ldots, 7)$ and final conditions $x_n = x_{n1}$, $n = 1, \ldots, l$, and $l < 7$ are given. It is required to find the time histories of m, α, and u_i such that x_i would satisfy the equations of motion, the preceding control constraints, the initial and final conditions, and minimize the given

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